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Existence and stability of travelling waves in (1 + 1) dimensions

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Abstract. We study the existence and stability of kinks and bell-shaped solitary travelling waves in $(1+1)$ dimensions. We prove that kinks are stable and bells are unstable; this yields a one-to-one correspondence between Liapunov, energetic and linear stability of travelling waves. Numerical examples are presented as an illustration of the above results.

1. Introduction

In this paper we study the existence and stability of travelling waves of the nonlinear wave equation (NLWE)

$$\phi_{tt} - \phi_{xx} - g(\phi) = 0. \qquad (1)$$

By a travelling wave, we mean a solution of the form

$$\phi(x, t) = \phi_v(\xi) \qquad \xi = x - vt$$

with $-1 < v < 1$ and ϕ_v a real-valued function defined on the entire real axis. We will be concerned with two types of travelling wave solutions.

(a) Bell solutions, or non-topological solitary waves. They correspond to solutions of equation (1) with symmetric boundary conditions at infinity, i.e.

$$\phi_v(\xi) \to a \qquad \text{as } \xi \to \pm\infty.$$

For a more detailed discussion see Magyari and Thomas (1984).

(b) Kink solutions, or topological solitary waves, which have different boundary conditions, i.e.

$$\phi_v(\xi) \to a_\pm \qquad \text{as } \xi \to \pm\infty$$

with $a_+ \neq a_-$.

In general, (1) does not have bell or kink solutions for an arbitrary function g. Berestycki and Lions (1983) found necessary and sufficient conditions on g for the existence of bell solutions.

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Among the solutions of (1), the most relevant from the physical point of view are those which are stable. In studying the stability of a solution, one has to specify the criterion used to define stability. There are several of them in the literature (see Vázquez 1982); here we mention three.

(I) Liapunov stability. When any initial data of the wave equation close to a travelling wave ϕ_v (with respect to a specified metric in the corresponding functional space) give rise to configurations which are also close to ϕ_v we will say that the solution is stable in the Liapunov sense. As the NLWE is invariant under the group of translations $\xi \to \xi_0$, $\phi_v(\xi - \xi_0)$ is also a solution for any $\xi_0 \in \mathbb{R}$, and the study of Liapunov stability must be done in this set of solutions, known as the ϕ_v orbit.

(II) Energetic stability. A travelling wave solution can be considered as a critical point of the energy subject to the constrain of constant momentum. A travelling wave is called energetic stable if it is a local minimum of the energy when the momentum is constant.

(III) Linear dynamical stability. A solution is dynamically stable if small perturbations do not destroy it. In this case we have to study the behaviour of the solution

$$\phi(x, t) = \phi_v(x - vt) + w(x - vt, t)$$

where the first-order approximation leads to a linear evolution equation for $w(\xi, t)$. We will say that ϕ_v is dynamically stable if $|w(\xi, t)|$ remains bounded for all t.

Sometimes a weaker requirement is also used. The invariance under translations of the nonlinear problem makes some solutions of the corresponding linear equation grow polynomially in time (zero mode). When all the solutions of the linearised equation exhibit no exponential growth we will say that ϕ_v is stable. Alternatively, ϕ_v is said to be dynamically unstable if there exist solutions w to the linear equation growing exponentially in t. The relationship between these three concepts of stability is still an open problem even for systems with a finite number of degrees of freedom.

The Liapunov and energetic stabilities in the case of travelling waves for the NLWE have been studied by Henry *et al* (1982) and Zhidkov and Kircher (1985), but the most fundamental results can be found in a paper by Grillakis *et al* (1987).

Equation (1) can be written as the following Hamiltonian system:

$$\frac{d\boldsymbol{u}}{dt} = JE'(\boldsymbol{u}) \tag{2}$$

where

$$\boldsymbol{u} = \begin{pmatrix} \phi \\ \phi t \end{pmatrix} \qquad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and $E'(\boldsymbol{u})$ is the functional derivative of the energy E, defined as

$$E(\boldsymbol{u}) = \int_{\mathbb{R}} dx[\tfrac{1}{2}(|\phi_t|^2 + |\phi_x|^2) + G(\phi)] \tag{3}$$

with $G(t) = \int_0^t ds\, g(s)$.

Since the NLWE is invariant under space translations, there exists a second conserved quantity, the momentum P:

$$P(\boldsymbol{u}) = \int_{\mathbb{R}} dx \phi_x \phi_t \equiv \tfrac{1}{2}\langle B\boldsymbol{u}, \boldsymbol{u}\rangle_{L^2 \oplus L^2} \tag{4}$$

where

$$B=\begin{pmatrix} 0 & -\partial_x \\ \partial_x & 0 \end{pmatrix}.$$

For the stability analysis of (2), Grillakis *et al* (1987) considered the linearised operator

$$H_v \equiv E''(\boldsymbol{u}_v)-vP''(\boldsymbol{u}_v)=\begin{pmatrix} L_v & v\partial_x \\ -v\partial_x & 1 \end{pmatrix} \tag{5}$$

where $\boldsymbol{u}_v=(\phi_v, v\phi_v)^{\mathrm{T}}$ and

$$L_v=-(1-v^2)\partial_x^2-g'(\phi_v). \tag{6}$$

They obtained the following results.

(I) Bell solutions. H_v has exactly one negative eigenvalue, its kernel is spanned by

$$\begin{pmatrix} \partial_x & 0 \\ 0 & \partial_x \end{pmatrix}\boldsymbol{u}_v$$

(coming from translation invariance) and its positive spectrum is bounded away from zero. Bell solutions are unstable in the sense of Liapunov stability and they are not local minima of the energy subject to constant momentum.

(II) Kink solutions. As in the previous case, the kernel of H_v is only generated by the translation invariance of the original nonlinear system, being the lowest eigenfunction of H_v. The positive spectrum is bounded away from zero. Kinks are stable in the Liapunov and energetic senses.

These results are special cases of a more general theory which says that under certain assumptions on the spectrum of the linearised operator (as verified in the above cases), there is a one-to-one correspondence between Liapunov stability and the minimisation of the energy E for the considered bound state when the second conserved quantity is held fixed.

In this paper we shall study the linear dynamical stability of ϕ_v by considering the linearised (about $\boldsymbol{u}$) evolution equation

$$\frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t}=JH_v(\boldsymbol{u}_v)\boldsymbol{w} \tag{7}$$

with $\boldsymbol{w}=(u, u_t)^{\mathrm{T}}$. We shall require $\boldsymbol{w}\in H^1(\mathbb{R})\oplus L^2(\mathbb{R})$ in order to have perturbations with finite energy and finite momentum.

We want to remark that up to now a systematic study of the linear stability of the NLWE has been only made for solutions which factorise the space and time dependence, i.e. $\boldsymbol{w}=\boldsymbol{w}(x)\mathrm{e}^{\lambda t}$, which is equivalent to the determination of the spectrum of H_v. Here we study the full evolution for any perturbation $\boldsymbol{w}$ with the only restriction that $\boldsymbol{w}\in H^1(\mathbb{R})\oplus L^2(\mathbb{R})$.

Our main results are as follows.

(I) Bells are dynamically unstable. More precisely, there exist solutions which grow exponentially in time.

(II) Kinks are dynamically stable. To the best of our knowledge, this result is new. We would like to show that it does not depend on the integrability of the systems. The proof of these statements will be given by constructing invariant subspaces for the linear evolution equation on which one could control the perturbation $\boldsymbol{w}$. This method is due to Weinstein (1985), who studied the linear stability of bound states of nonlinear Schrödinger equations.

Our results give a one-to-one correspondence between the stability of the nonlinear system and the stability of the linear equation in some sense. Physically this correspondence relies heavily on the fact that the 'kinetic energy' $\int_{\mathbb{R}} dx|\phi_t|^2$ is positive definite and should break down if this property does not hold. The latter is already known for systems with a finite number of degrees of freedom and also for spinor field models, as can be seen in Blanchard *et al* (1987).

The paper is organised as follows. The results about necessary and sufficient conditions for the existence of kink solutions are given in section 2. In section 3 we study the linear dynamical stability of such solutions as well as bell solutions. Finally, in section 4 we illustrate our results with some examples.

2. Existence of travelling waves

Finding the travelling waves ϕ_v of the NLWE amounts to solving the ordinary differential equation:

$$-(1-v^2)\phi_v'' = g(\phi_v). \tag{8}$$

It is enough to study solutions in their rest frame, i.e. $v=0$, because the corresponding solution with $v \neq 0$ can be easily obtained by rescaling (Lorentz transformation)

$$\phi_v(\xi) = \phi_0\left(\frac{\xi}{\sqrt{(1-v^2)}}\right).$$

Thus, in all that follows we will consider only solutions having zero velocity. We will also assume that g is a continuously differentiable function from $\mathbb{R}$ into $\mathbb{R}$ (in fact continuous locally Lipschitz would be enough for the existence).

For bell solutions we may choose the boundary condition at infinity as zero and then assume that g is an odd function.

The existence result for bell solutions given by Berestycki and Lions (1983) is the following. Consider the problem

$$\begin{aligned} &-\phi'' = g(\phi) && \phi \in C^2(\mathbb{R}) \\ &\lim_{x\to\pm\infty} \phi(x) = 0 && \phi(x_0) > 0 \end{aligned} \tag{9}$$

for some $x_0 \in \mathbb{R}$. Then the following theorem holds.

Theorem 2.1. A necessary and sufficient condition for the existence of a solution of problem (9) is that

$$z_0 = \inf\{z>0;\ G(z)=0\} \tag{10}$$

exists and $g(z_0)>0$, with $G(z)$ defined by $G(z) = \int_0^z ds\, g(s)$.

If (10) is satisfied, (9) has a unique solution up to translations of the origin, and this solution satisfies (after a suitable translation of the origin):

$$\phi(x) = \phi(-x) \qquad x \in \mathbb{R} \tag{11a}$$

$$\phi(x) > 0 \qquad x \in \mathbb{R} \tag{11b}$$

$$\phi(0) = z_0 \tag{11c}$$

$$\phi'(x) < 0 \qquad x > 0. \tag{11d}$$

Following the lines of this theorem, a similar result can be stated for kink solutions. We will assume $a_+ = -a_-$ without loss of generality and that g is again an odd function. In addition, we will require $g'(0) > 0$.

Consider the following Neumann boundary value problem:

$$-\phi'' = g(\phi) \qquad \phi \in C^2(\mathbb{R}) \qquad \lim_{x\to\pm\infty} \phi'(x) = 0 \tag{12}$$

with $\phi'(x_0) > 0$ and $\phi(x_0) = 0$ for some $x_0 \in \mathbb{R}$.

Theorem 2.2. A necessary and sufficient condition for the existence of a solution of problem (12) is that

$$p_0 = \inf\{p > 0;\ g(p) = 0\} \tag{13}$$

exists, and $G(p_0) > 0$.

If (13) is satisfied there exists a solution which after a suitable translation of the origin satisfies

$$\phi(-x) = -\phi(x) \qquad x \in \mathbb{R} \tag{14a}$$

$$\phi'(x) > 0 \qquad x \in \mathbb{R} \tag{14b}$$

$$\phi'(0) = \sqrt{2G(p_0)} \tag{14c}$$

$$\lim_{x\to\pm\infty} \phi(x) = \pm p_0. \tag{14d}$$

Moreover, ϕ is unique (up to translations) in the sense that there is no different solution satisfying (14d).

Proof. We can obtain ϕ as the solution of an initial value problem for the differential equation (12) with initial conditions $\phi(0) = 0$ and $\phi'(0) = \sqrt{2G(p_0)} > 0$. This solution exists and is unique on a maximal interval $(-\bar{x}, \bar{x})$ satisfying $\phi(-x) = -\phi(x)$. We have also $\phi'(x) > 0$ in $(-\bar{x}, \bar{x})$. Indeed, let $\phi'(x_0) = 0$ and x_0 be the first zero of ϕ'; then $G(\phi(x_0)) = G(p_0)$, and consequently $\phi(x_0) = p_0$ which implies $\phi \equiv p_0$, but this is incompatible with $\phi(-x) = -\phi(x)$.

Obviously ϕ is bounded, and by standard continuation arguments it is defined on the whole real axis. The solution ϕ also satisfies the asymptotic conditions

$$\lim_{x\to\infty} \phi(x) = p_0 \qquad \text{and} \qquad \lim_{x\to\infty} \phi'(x) = 0.$$

Let us go on to prove that ϕ is unique up to translations in the class of solutions satisfying $\lim_{x\to\infty}\phi = p_0$.

Let ψ be another solution of (12) satisfying $\lim_{x\to\infty} \psi = p_0$. After translation, if necessary, we have $\psi(0) = 0$ and $\frac{1}{2}\psi'^2(0) = G(p_0)$. By the uniqueness of the initial value problem we conclude that $\phi = \psi$.

Finally, we will show that condition (13) is necessary. According to the conservation law

$$\tfrac{1}{2}\phi'^2(x) + G(\phi(x)) = \tfrac{1}{2}\phi'^2(0)$$

ϕ can be bounded by

$$\phi(x) \leq G^{-1}(\tfrac{1}{2}\phi'^2(0)).$$

This property, together with $\phi'(x)>0$ for $x>0$, implies that $\phi''(x)\leq 0$ for $x>0$, and consequently that the $\lim_{x\to\infty}\phi(x)\equiv L$ exists. It can be inferred from here that $g(L)=0$ and $G(L)>0$, contradicting the assumption that no such number exits.

The case $p_0>0$ and $G(p_0)\leq 0$ is impossible because of the condition $g'(0)>0$. □

3. Linear dynamical stability

In this section we study the evolution of the linear stability equation (7). Since it is enough to analyse the case $v=0$, we consider the linear problem

$$\frac{\mathrm{d}w}{\mathrm{d}t}=JH_0(u_0)w \qquad w(0)=w_0 \tag{15}$$

where $w_0\in X\equiv H^1(\mathbb{R})\oplus L^2(\mathbb{R})$,

$$H_0=\begin{pmatrix} L_0 & 0\\ 0 & 1\end{pmatrix} \tag{16}$$

$$L_0=-\partial_x^2-g'(\phi_0). \tag{17}$$

(a) Bell solutions. We are going to explicitly construct a solution $w\in X$ which grows exponentially in time. It is known (see Grillakis *et al* (1987), for instance) that L_0 has exactly one strictly negative eigenvalue $-\alpha_0^2$ $(\alpha_0>0)$ with eigenfunction X_0

$$L_0X_0=-\alpha_0^2X_0 \qquad X_0\in H^1(\mathbb{R}).$$

If we define

$$w(t)=\mathrm{e}^{\alpha_0 t}\begin{pmatrix} X_0\\ \alpha_0X_0\end{pmatrix} \tag{18}$$

then $w(t)$ is a solution of (15) with initial value

$$w_0(t)=\begin{pmatrix} X_0\\ \alpha_0X_0\end{pmatrix}\in X. \tag{19}$$

From here $\|w(t)\|_X=c\mathrm{e}^{\alpha_0 t}$, and we can establish the next theorem.

Theorem 3.1. The bell solutions of the NLWE are unstable in the sense of linear dynamical stability.

Remark. Our result concerning solitary waves when $\partial_x\phi$ has at least a zero is consistent with the numerical experiments about the collisions of ϕ^4 solitary waves by Campbell and Peyrard (1986) in the following sense. If we assume that there are solitary waves of ϕ^4 such that $\partial_x\phi$ has at least one zero, they will be dynamically unstable. Furthermore, if they exist they are bound states (without internal degrees of freedom) of the associated kinks and antikinks and even a small amount of radiation destroys such structure.

(b) Kink solutions. Following the idea of Weinstein (1985), we construct a subspace M of $X=H^1(\mathbb{R})\oplus L^2(\mathbb{R})$ which is orthogonal to the directions governed by the invariance of the nonlinear system. This subspace is defined as

$$M=X\cap[N_g((JH_0)^*)]^\perp \tag{20}$$

where $(JH_0)^*$ is the adjoint of JH_0 and $N_g(A)=\bigcup_{n=1}^{\infty}N(A^n)$ is the generalised nullspace of the operator A. By $\perp$ we denote the orthogonal with respect to the inner product of $Y\equiv L^2(\mathbb{R})\oplus L^2(\mathbb{R})$.

First of all, let us determine the form of the elements of the spaces $N_g(JH_0)$ and $N_g((JH_0)^*)$.

Proposition 3.2. The spaces $N_g(JH_0)$ and $N_g((JH_0)^*)$ can be written as

$$N_g(JH_0)=N((JH_0)^2)\cup N(JH_0) \qquad N_g((JH_0)^*)=N((JH_0)^{*2})\cup N((JH_0)^*)$$

and are spanned by the biorthogonal set $\{e_1, e_2\}$, $\{f_1, f_2\}$ defined as

$$e_1=\begin{pmatrix}\partial_x\phi_0\\0\end{pmatrix} \qquad e_2=\begin{pmatrix}0\\\partial_x\phi_0\end{pmatrix} \tag{21a}$$

$$f_1=\begin{pmatrix}\partial_x\phi_0\\0\end{pmatrix} \qquad f_2=\begin{pmatrix}0\\\partial_x\phi_0\end{pmatrix} \tag{21b}$$

where

$$(f_i, e_j)_Y=\delta_{ij}\int_{\mathbb{R}}\mathrm{d}x|\partial_x\phi_0|^2.$$

Proof. For kinks, the ground state of L_0 is given by $\partial_x\phi_0$ and $L_0(\partial_x\phi_0)=0$, where L_0 is non-negative (see Grillakis *et al* (1987) for instance). As

$$JH_0=\begin{pmatrix}0 & 1\\-L_0 & 0\end{pmatrix}$$

the expression $JH_0u=0$ implies that $u\sim e_1$, and consequently $N(JH_0)=\langle e_1\rangle$. From

$$(JH_0)^2=\begin{pmatrix}-L_0 & 0\\0 & -L_0\end{pmatrix}$$

we have that $N((JH_0)^2)=\langle e_1, e_2\rangle$. The equation $(JH_0)^3u=0$ leads to the equations $-L_0u_2=0$ and $L_0^2u_1=0$. The first one implies that $u_2\sim\partial_x\phi_0$ and the second one that $L_0u_1=\lambda\partial_x\phi_0$. From the inner product

$$0=(\partial_x\phi_0, L_0u_1)_{L^2}=\lambda(\partial_x\phi_0, \partial_x\phi_0)_{L^2}$$

we can conclude that $\lambda=0$ and obtain finally that $u_1\sim\partial_x\phi_0$ and $N((JH_0)^3)=N((JH_0)^2)$, which proves the first part of the proposition.

Repeating the same calculation for

$$(JH_0)^*=\begin{pmatrix}0 & -L_0\\1 & 0\end{pmatrix}$$

one can obtain the corresponding result for $N_g((JH_0)^*)$. The biorthogonality condition is obvious from here. □

The biorthogonality of $N_g(JH_0)$ and $N_g((JH_0)^*)$ allows us to write

$$X=M\oplus N_g(JH_0). \tag{22}$$

The evolution in $N_g(JH_0)$ is described in the following proposition.

Proposition 3.3. Let $w(t)$ be a solution of (15) with $w_0 \in N_g(JH_0)$. Then $w(t) \in N_g(JH_0)$ for all t and

$$w(t) = [(w(t), f_1)_Y e_1 + (w(t), f_2)_Y e_2] \int_{\mathbb{R}} \mathrm{d}x (\partial_x \phi_0)^2 \tag{23}$$

where

$$(w(t), f_1)_Y = (w_0, f_1)_Y \tag{24a}$$

$$(w(t), f_2)_Y = (w_0, f_2)_Y t + (w_0, f_1)_Y. \tag{24b}$$

Proof. The representation (23) follows directly from the biorthogonality of sets $\{e_1, e_2\}$ and $\{f_1, f_2\}$. Writing $w(t)$ as

$$w(t) = c_1(t) e_1 + c_2(t) e_2$$

and inserting this expression into (15), we obtain

$$\dot{c}_1(t) e_1 + \dot{c}_2 e_2 = c_2(t) e_1$$

that implies $c_2(t) = c_2(0)$ and $c_1(t) = c_2(0) t + c_1(0)$, which are precisely (24*a*) and (24*b*). □

It is now clear that if w_0 has a vanishing component in M, this component must be $w(t)$.

As a consequence of proposition 3.3 we have the corollary below.

Corollary 3.4. M is an invariant subspace for $\Omega(t) = \exp(tJH)$.

It is easy to check that the quadratic form Q defined by

$$Q(w) = \tfrac{1}{2}(H_0 w, w)_Y \tag{25}$$

is a conserved quantity for the linear evolution problem (15).

Let us now show that the restriction on M of the quadratic form Q defines a norm which is equivalent to the norm on X defined by

$$\|w\|_X = \|w_1\|_{H^1} + \|w_2\|_{L^2}.$$

Proposition 3.5. There exist constants k_1, k_2 such that

$$k_1 \|w\|_X^2 \leqslant Q(w) \leqslant k_2 \|w\|_X^2 \tag{26}$$

for any $w \in M$.

Proof. From equation (25) we have that

$$Q(w) = \tfrac{1}{2}(L_0 w_1, w_1)_{L^2} + \tfrac{1}{2}(w_2, w_2)_{L^2}.$$

The existence of the upper bound is obvious. For the lower bound we use the fact that $w \in M$ implies $(w_1, \partial_x \phi_0)_{L^2} = 0$, but L_0 is strictly positive on the orthogonal complement of its kernel, i.e.

$$(L_0 w_1, w_1)_{L^2} \geqslant \varepsilon (w_1, w_1)_{H^1}$$

for some $\varepsilon > 0$, which implies the existence of the lower bound. □

An easy consequence of proposition 3.5 is our following stability result for kinks.

Theorem 3.6. Kinks are always stable in the sense of linear dynamical stability.

Proof. Proposition 3.5 tell us that for any initial value $w_0 \in M$ the solution $w(t)$ can be bounded as

$$\|w(t)\|_X \leq c\|w_0\|_X \tag{27}$$

for some $c > 0$.

From here it follows that $\|w_1(t)\|_{H^1} \leq c\|w_0\|_X$. Since w_1 is bounded in $H^1(\mathbb{R})$ for all t, it is also bounded in L^∞ for all t and the solutions in the complementing space grow at most linearly in time.

4. Numerical results

In this section we show some numerical results related to the stability of the kinks associated with the wave equation

$$\phi_{tt} - \phi_{xx} - m^2\phi + \lambda\phi^3 = 0. \tag{28}$$

Without loss of generality we can take $m = 1$ and $\lambda = 1$. The numerical integration is accomplished with the finite-difference scheme

$$\begin{gathered}\frac{\phi_j^{n+1} - 2\phi_j^n + \phi_j^{n-1}}{\Delta t^2} - \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2} + \frac{G(\phi_j^{n+1}) - G(\phi_j^{n-1})}{\phi_j^{n+1} - \phi_j^{n-1}} = 0 \\ G(x) \equiv \tfrac{1}{4}(x^2 - 1)^2\end{gathered} \tag{29}$$

which has a conserved discrete energy given by

$$\begin{aligned}E^n \equiv \sum_{j=-\infty}^{+\infty} \Delta x\Bigg[&\frac{1}{2}\left(\frac{\phi_j^n - \phi_j^{n-1}}{\Delta t}\right)^2 + \frac{1}{2}\left(\frac{\phi_j^n - \phi_{j-1}^n}{\Delta x}\right)\left(\frac{\phi_j^{n-1} - \phi_{j-1}^{n-1}}{\Delta x}\right) \\ &+ \frac{1}{2}\left(G(\phi_j^n) + G(\phi_j^{n-1})\right)\Bigg].\end{aligned} \tag{30}$$

The main properties of this scheme are well described in Pascual and Vázquez (1985) and references therein.

We show in figures 1-5 the evolution of the initial data

$$\phi(x, 0) = \tanh\frac{x}{\sqrt{2}} + \eta(x) \qquad \phi_t(x, 0) = 0 \tag{31}$$

where

$$\eta(x) = \begin{cases} a\exp\left(-\dfrac{kx^2}{b^2 - x^2}\right) & x \in [-b, b] \quad k > 0 \\ 0 & x \notin [-b, b]. \end{cases} \tag{32}$$

Equation (31) represents a kink initially at rest with a small deformation. We also present the evolution of the energy density of the perturbed kink. In all the figures $a = 0.2$, $b = 5$ and $k = 3.5$.

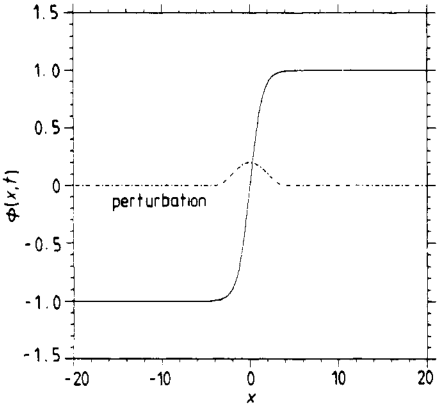

Figure 1. Representation of the unperturbed solution $\phi(x,0) = \tanh(x/\sqrt{2})$. The chain curve corresponds to the introduced perturbation.

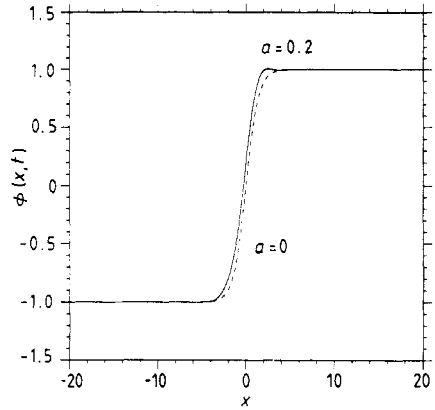

Figure 2. Representation of the initial data $\phi(x,0) = \tanh(x/\sqrt{2}) + \eta(x)$ for $a = 0.2$. This graph is compared with the unperturbed solution (broken curve).

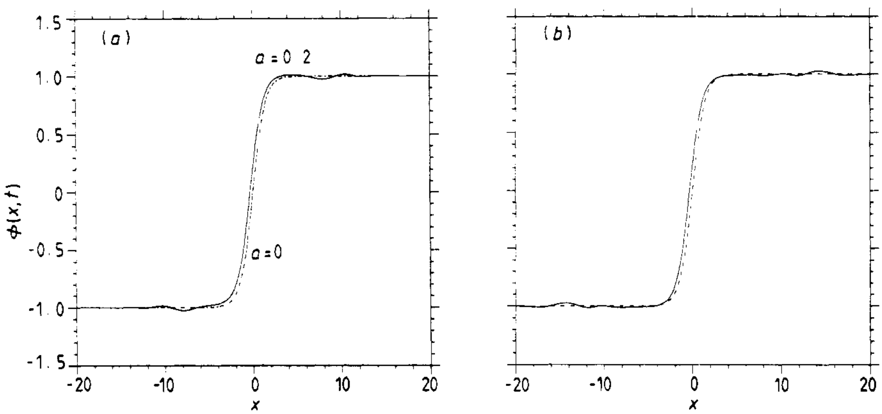

Figure 3. Evolution representation of the kink at (a) $t = 500\Delta t$ and (b) $t = 1500\Delta t$. In all cases $\Delta t = 0.025$. These figures show the effect of the perturbation on the initial function as a small group of travelling waves that do not destroy the initial wave shape, which proves its stability.

It can be seen from figures 1-5 that the kink structure is preserved under the small perturbation (32), in agreement with our mathematical results.

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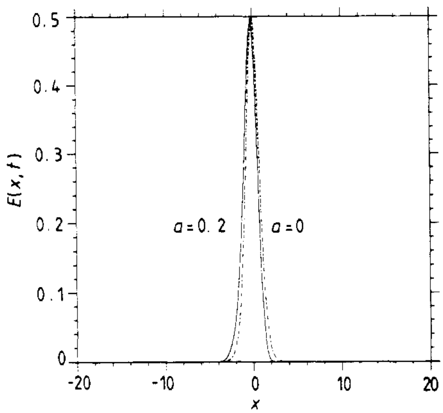

Figure 4. Energy density against x at $t=0$ for $a=0$ (chain curve) and $a=0.2$. The total energy, measured by the area under the curve, is $E=0.942\,77$ ($a=0$) and $E=0.955\,55$ ($a=0.2$). The theoretical value for the first case is $E_{\text{theor}}=4/3\sqrt{2}\approx 0.942\,81$.

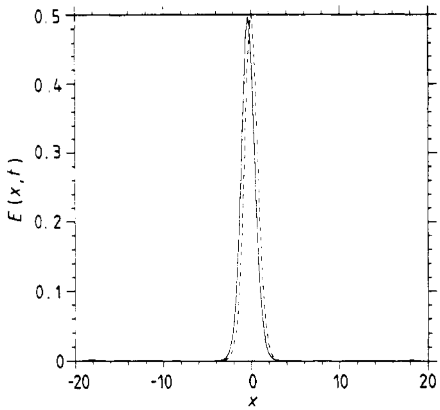

Figure 5. Energy density against x at $t=1500\Delta t$ compared with the unperturbed density (broken curve). The total energy is a conserved quantity so $E=0.955\,55$ for $a=0.2$.

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